TUTORATO ANALISI 1 – 06/04/2020

$$\int_0^{\pi/4} (\cos^2 x + 1)\tan x \, dx \underset{\substack{y=\cos x \\ dy = -\sin x\, dx}}{=} -\int_1^{\sqrt{2}/2} \frac{(y^2+1)}{y}\, dy =$$

$$= \int_{\sqrt{2}/2}^{1} y\, dy + \int_{\sqrt{2}/2}^{1} \frac{1}{y}\, dy = \left[y^2/2 \right]_{\sqrt{2}/2}^{1} + \left[\log y \right]_{\sqrt{2}/2}^{1} =$$

$$= \frac{1}{2} - \frac{1}{4} + 0 - \log\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{4} - \log \frac{\sqrt{2}}{2}$$

$$\int_{-1}^{+1} \frac{2^x}{4 - |1-2^x|(1-2^x)}\, dx \underset{\substack{y=2^x \\ dy = \log 2 \cdot 2^x dx}}{=} \int_{1/2}^{2} \frac{1}{\log 2} \frac{1}{4 - |1-y|(1-y)}\, dy =$$

$$= \underbrace{\int_{1/2}^{1} \frac{1}{\log 2} \frac{1}{4-(1-y)^2}\, dy}_{①} + \underbrace{\int_{1}^{2} \frac{1}{\log 2} \frac{1}{4+(1-y)^2}\, dy}_{②} =$$

① $$\frac{1}{\log 2} \int_{1/2}^{1} \frac{1}{\underbrace{(2-(1-y))(2+(1-y))}}\, dy$$

$$\frac{A}{1+y} + \frac{B}{3-y}$$

$$A(3-y) + B(1+y) = 1$$

$$\Rightarrow \begin{cases} -Ay + By = 0 \\ 3A + B = 1 \end{cases} \Rightarrow \begin{cases} A = B \\ 4A = 1 \end{cases} \qquad A = B = 1/4$$

$$= \frac{1}{\log 2} \int_{1/2}^{1} \frac{1}{4} \cdot \frac{1}{1+y}\, dy + \frac{1}{\log 2} \int_{1/2}^{1} \frac{1}{4} \frac{1}{3-y}\, dy =$$

$$= \frac{1}{4 \log 2} \Big[\log(1+y) \Big]_{1/2}^{1} + \frac{1}{4 \log 2} \Big[-\log(3-y) \Big]_{1/2}^{1}$$

$$= \frac{1}{4 \log 2} \Big[\cancel{\log 2} - \log(3/2) \Big] + \frac{1}{4 \log 2} \left(-\cancel{\log 2} + \log 5/2 \right)$$

$$= -\frac{\log 3}{4 \log 2} + \cancel{\frac{\log 2}{4 \log 2}} + \frac{\log 5}{4 \log 2} - \cancel{\frac{\log 2}{4 \log 2}}$$

(2) $$\int_{1}^{2} \frac{1}{\log 2} \underbrace{\frac{1}{4 + (1-y)^2}} dy = \int_{1}^{2} \frac{1}{\log 2} \frac{1}{4} \underbrace{\frac{1}{1 + \left(\frac{1-y}{2}\right)^2}} dy =$$

$$= \frac{1}{4 \log 2} \left[-2 \arctan\left(\frac{1-y}{2} \right) \right]_{1}^{2} =$$

$$= \frac{1}{4 \log 2} \left(-2 \arctan\left(-\frac{1}{2} \right) + 2 \arctan 0 \right) =$$

$$= \frac{1}{\ } \arctan(1/2)$$

$2\log 2$

$$① + ② \quad \frac{1}{4\log 2}\left(-\log 3 + \log 5\right) + \frac{1}{2\log 2}\arctan\left(\tfrac{1}{2}\right)$$

Convergenza

$$\int_0^{+\infty} \frac{2^x}{4 - |1-2^x|(1-2^x)}\,dx = \int_0^{+\infty} \frac{2^x}{4+(1-2^x)^2}\,dx$$

$$\int_0^R \frac{2^x}{4+(1-2^x)^2}\,dx \underset{y=2^x}{=} \int_1^{2^R} \frac{1}{\log 2}\,\frac{1}{4+(1-y)^2}\,dy =$$

$$dy = \log 2 \cdot 2^x dx$$

$$= \frac{1}{\log 2}\cdot\frac{1}{4}\int_1^{2^R} \frac{1}{\left(1+\left(\frac{1-y}{2}\right)^2\right)}\,dy = \frac{1}{4\log 2}\left[-2\arctan\left(\frac{1-y}{2}\right)\right]_1^{2^R}$$

$$= \frac{1}{2\log 2}\cdot\left(-\arctan\left(\frac{1-2^R}{2}\right)\right) \longrightarrow \frac{1}{2\log 2}\cdot\frac{\pi}{2}$$

Per quali $\alpha > 0$

$$\int_0^{+\infty} \frac{e^{-\sqrt{x}}}{x^\alpha}\,dx \qquad \text{converge?}$$

① $\displaystyle\int_0^1 \frac{e^{-\sqrt{x}}}{x^\alpha}\,dx$

$$\int_0^1 \frac{1}{e}\frac{1}{x^\alpha}\,dx \leq \int_0^1 \frac{e^{-\sqrt{x}}}{x^\alpha}\,dx \leq \int_0^1 \frac{1}{x^\alpha}\,dx$$

$\displaystyle\int_0^1 \frac{1}{x^\alpha}\,dx$ converge se $\alpha < 1$

② $\displaystyle\int_1^{+\infty} \frac{e^{-\sqrt{x}}}{x^\alpha}\,dx$

$k > 0$ so che $e^x \geq x^k$ $x > 0$

$e^{-x} \leq \dfrac{1}{x^k}$ $\forall k > 0$

$\rightsquigarrow$ $e^{-\sqrt{x}} \leq \dfrac{1}{x^{k/2}}$

$$0 \leq \frac{e^{-\sqrt{x}}}{x^\alpha} \leq \frac{1}{x^{k/2} \cdot x^\alpha} \qquad \alpha + \frac{k}{2} > 1$$

$k > 2 - 2\alpha$

$\Rightarrow \displaystyle\int_1^{+\infty} \frac{e^{-\sqrt{x}}}{x^\alpha}\,dx$ converge

$$\int_1^2 \frac{1}{\sqrt{x^\alpha - 1}}\, dx$$

$\alpha > 0$ dire per quali α converge

$$y = x - 1 \quad \rightsquigarrow \quad x = y + 1$$

$$\int_0^1 \frac{1}{\sqrt{(y+1)^\alpha - 1}}\, dy$$

$$\lim_{y \to 0^+} \frac{y^{\alpha/2}}{\sqrt{(y+1)^\alpha - 1}}$$

$$\sqrt{(y+1)^\alpha - 1} =$$

$$\frac{1}{2\sqrt{(y+1)^\alpha - 1}} \cdot \alpha (y+1)^{\alpha - 1}$$

$$\lim_{y \to 0^+} \frac{y^\alpha}{(y+1)^\alpha - 1} = \lim_{y \to 0^+} \frac{y^\alpha}{y^\alpha} \cdot \frac{1}{(1 + 1/y)^\alpha - 1} = 0$$

$\alpha \neq 1$

$$x^\alpha - 1 = \alpha (x - 1) + \frac{\alpha(\alpha - 1)}{2} (x - 1)^2 + O(x^2)$$

$$(x^\alpha - 1)' = \alpha x^{\alpha - 1}$$

$$\frac{1}{\sqrt{x^{\alpha}-1}} \sim \frac{1}{\sqrt{x-1}} \quad ✓$$

$$\int_1^2 \frac{1}{\sqrt{x-1}}\,dx \qquad \text{converge}$$

$$\int_1^2 \frac{1}{\sqrt{x^{\alpha}-1}}\,dx \qquad \text{converge } \forall\, \alpha > 0$$

Sia $f:[a,b] \to \mathbb{R}$ derivabile ($f\ C^1$) t.c. f' e' integrabile. Dire che

$$\lim_{n\to+\infty} \int_a^b f(x)\sin(nx)\,dx = 0$$

$$\int_a^b f(x)\sin(nx)\,dx = \left[-f(x)\,\frac{\cos(nx)}{n}\right]_a^b + \int_a^b f'(x)\,\frac{\cos(nx)}{n}\,dx =$$

$$= \underbrace{-f(b)\cdot\frac{\cos(nb)}{n} + f(a)\,\frac{\cos(na)}{n}} + \frac{1}{n}\int_a^b f'(x)\cos(nx)\,dx$$

$$\left|\int^b \qquad \right|$$

$$\left|\int_a f(x)\sin(nx)\,dx\right| \le \frac{1}{n}|f(b)| + \frac{1}{n}|f(a)| +$$

$$+\frac{1}{n}\left|\int_a^b f'(x)\cos(nx)\,dx\right| \le (*)$$

$$\int_a^b |f'(x)|\,dx \quad < +\infty$$

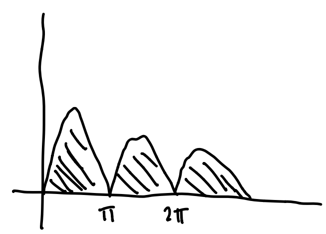
$$(*) \quad \frac{1}{n}|f(b)| + \frac{1}{n}|f(a)| + \frac{1}{n}(b-a)\max f'$$

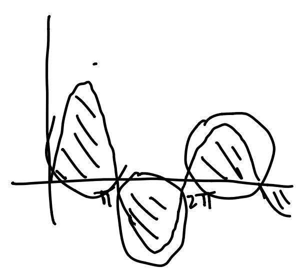
$f : [0, +\infty) \to \mathbb{R}$ $\quad C^1 \quad$ monotona deb. decr.

$f \geq 0 \qquad \lim_{x\to+\infty} f(x) = 0$

$$\int_0^{+\infty} f(x)\sin(x)\,dx \quad \text{converge}$$

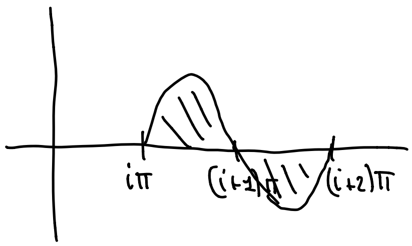
$$\int_0^{+\infty} f(x)|\sin x|\,dx \quad \text{NO}$$





$$\int_0^{+\infty} f(x)\sin x\,dx = \sum_{i=0}^{+\infty} \int_{i\pi}^{(i+1)\pi} f(x)\sin x\,dx$$

$$\left| \int_{i\pi}^{(i+1)\pi} f(x) \sin x \, dx \right| \geq \left| \int_{(i+1)\pi}^{(i+2)\pi} f(x) \sin x \, dx \right|$$



Per Leibniz

Vale $\displaystyle\sum_{i=0}^{+\infty} \int_{i\pi}^{(i+1)\pi} f(x) \sin x \, dx$ converge e quindi anche $\displaystyle\int_0^{+\infty} f(x) \sin x \, dx$ converge.